

ON ASYMPTOTIC STABILITY OF STANDING WAVES OF DISCRETE SCHRÖDINGER EQUATION IN \mathbb{Z}

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ABSTRACT. We prove an analogue of a classical asymptotic stability result of standing waves of the Schrödinger equation originating in work by Soffer and Weinstein. Specifically, our result is a transposition on the lattice \mathbb{Z} of a result by Mizumachi [M1] and it involves a discrete Schrödinger operator $H = -\Delta + q$. The decay rates on the potential are less stringent than in [M1], since we require $q \in \ell^{1,1}$. We also prove $|e^{itH}(n, m)| \leq C\langle t \rangle^{-1/3}$ for a fixed C requiring, in analogy to Goldberg & Schlag [GSc], only $q \in \ell^{1,1}$ if H has no resonances and $q \in \ell^{1,2}$ if it has resonances. In this way we ease the hypotheses on H contained in Pelinovsky & Stefanov [PS], which have a similar dispersion estimate.

§1 INTRODUCTION

We consider the discrete Laplacian Δ in \mathbb{Z} defined by

$$(\Delta u)(n) = u(n+1) + u(n-1) - 2u(n).$$

In $\ell^2(\mathbb{Z})$ we have for the spectrum $\sigma(-\Delta) = [0, 4]$. Let for $\langle n \rangle = \sqrt{1+n^2}$

$$\begin{aligned} \ell^{p,\sigma}(\mathbb{Z}) &= \{u = \{u_n\} : \|u\|_{\ell^{p,\sigma}}^p = \sum_{n \in \mathbb{Z}} \langle n \rangle^{p\sigma} |u(n)|^p < \infty\} \text{ for } p \in [1, \infty) \\ \ell^{\infty,\sigma}(\mathbb{Z}) &= \{u = \{u(n)\} : \|u\|_{\ell^{\infty,\sigma}} = \sup_{n \in \mathbb{Z}} \langle n \rangle^\sigma |u(n)| < \infty\}. \end{aligned}$$

We will denote $\ell^{p,\sigma}(\mathbb{Z})$ with $\ell^{p,\sigma}$. We will set $\ell^p = \ell^{p,0}$. We will write $\ell^{p,\sigma}(\mathbb{Z}, \mathbb{R})$ when we restrict to functions such that $u_n \in \mathbb{R}$ for all n .

We consider a potential $q = \{q(n), n \in \mathbb{Z}\}$ with $q(n) \in \mathbb{R}$ for all n . We consider the discrete Schrödinger operator H

$$(1.1) \quad (Hu)(n) = -(\Delta u)(n) + q(n)u(n).$$

We assume:

(H1) $q \in \ell^{1,1}$.

- (H2) H is generic, in the sense of Lemma 5.3.
(H3) $\sigma_d(H)$ consists of exactly one eigenvalue $-E_0$, with $-E_0 \notin [0, 4]$.

By Lemma 5.3 below $\dim \ker(H + E_0) \leq 1$ in ℓ^2 . We denote by $\varphi_0(n)$ a generator of $\ker(H + E_0)$ normalized so that $\|\varphi_0\|_{\ell^2} = 1$. Consider now the discrete nonlinear Schrödinger equation (DNLS)

$$(1.2) \quad i\partial_t u(t, n) - (Hu)(t, n) + |u(t, n)|^6 u(t, n) = 0.$$

We look at a particular family of solutions $e^{i\omega t} \phi_\omega$ of (1.2), or equivalently of

$$(1.3) \quad (Hu)(n) - |u(n)|^6 u(n) = -\omega u(n).$$

By standard bifurcation arguments we have the following result, see Appendix A:

Lemma 1.1. *Assume (H1)–(H3). There is a family $\omega \rightarrow \phi_\omega$ of standing waves solving (1.3) with the following properties. For any $\sigma \geq 0$ there is an $\eta > 0$ such that $\omega \rightarrow \phi_\omega$ belongs to $C^\omega([E_0, E_0 + \eta[, \ell^{2,\sigma}) \cap C^0([E_0, E_0 + \eta[, \ell^{2,\sigma})$. We have $\phi_\omega(n) \in \mathbb{R}$ for any n and there are fixed $a > 0$ and $C > 0$ such that $|\phi_\omega(n)| \leq Ce^{-a|n|}$. As $\omega \rightarrow E_0$ we have in $C^\infty([E_0, E_0 + \eta[, \ell^{2,\sigma}) \cap C^0([E_0, E_0 + \eta[, \ell^{2,\sigma})$ the expansion*

$$(1.4) \quad \phi_\omega = (\omega - E_0)^{\frac{1}{6}} \|\varphi_0\|_{\ell^8}^{-\frac{4}{3}} (\varphi_0 + O(\omega - E_0)).$$

The main aim of this paper is the following asymptotic stability result:

Theorem 1.2. *Consider in Lemma 1.1 $\sigma > 0$ large and $\eta > 0$ small. Assume (H1)–(H3). For any $\omega_0 \in]E_0, E_0 + \eta[$ there exist an $\epsilon_0 > 0$ and a $C > 0$ such that if we pick $u_0 \in \ell^2$ with $\|u_0 - \phi_{\omega_0}\|_{\ell^2} < \epsilon < \epsilon_0$, then there exist $\omega_+ \in (E_0, E_0 + \eta_0)$, $\Theta \in C^1(\mathbb{R})$ and $u_+ \in \ell^2$ with $|\omega_+ - \omega_0| + \|u_+\|_{\ell^2} \leq C\epsilon$ such that if $u(t, n)$ is the corresponding solution of (1.2) with $u(0, n) = u_0(n)$, then*

$$\lim_{t \rightarrow \infty} \|u(t) - e^{i\Theta(t)} \phi_{\omega_+} - e^{it\Delta} u_+\|_{\ell^2} = 0.$$

For more precise statements see Theorem 4.1 and Lemma 4.4 in §4. Theorem 1.2 is related to [SW2]. The series [SW1-2] inspired a long list of papers on asymptotic stability of both large and small ground states in the continuous case, see [PW,BP1-2,SW3,Wd,C1,TY1-3,Ts,C2,BS,P,RSS,SW4,GNT,S,KS,GS1, C3,M1-2,GS2,CM,C4-5,KZ,CT,CV2]. We refer for a discussion of the state of the art to the introductions in [CM,C5]. In the case of the lattice \mathbb{Z} the results on dispersion in [SK,KKK,PS] are enough to develop an exactly analogous theory. The fact that the dispersion rate of $e^{it\Delta}$ in \mathbb{Z} is $\langle t \rangle^{-1/3}$, [SK], instead of $t^{-1/2}$ in \mathbb{R} , is analogous,

in fact easier, to the situation in [C3,CV2] which considers operators with potentials periodic in space. It is possible to develop in \mathbb{Z} a theory completely analogous to the one in \mathbb{R} , allowing H to have any finite number of eigenvalues, if, in analogy to the continuous case, we assume an important nonlinear hypothesis, the so called Fermi Golden Rule (FGR) in [CM,C5]. However, the fact that the spectrum $\sigma(-\Delta)$ is $[0, 4]$ in \mathbb{Z} instead of $[0, \infty)$ in \mathbb{R} , implies that there will be cases in \mathbb{Z} when certainly the FGR does not hold. In that case Theorem 1.2 does not hold any more, see [C6].

The theory can be developed also for large solitons, following the framework in [KS]. The fact of the slow $\langle t \rangle^{-1/3}$ dispersion rate is reflected in the fact that, since we use Strichartz like estimates, we can prove Theorem 1.2 with nonlinearity $|u|^{p-1}u$ for $p \geq 7$, rather than for $p \geq 5$ as in [M1,C4]. In (1.2) we choose nonlinearity $|u|^6u$ for definiteness, because what matters is to have a nonlinearity $\beta(|u|^2)u$ with $\beta(t) \in C^2$ with $\beta(0) = d\beta(0)/dt = d^2\beta(0)/dt^2 = 0$. Notice that the C^ω regularity proved in Lemma 1.1, which is used in [C6], is not necessary here: all we need is Lemma 1.1 with C^ω and C^∞ replaced by C^2 .

For the proof of the nonlinear estimates and the use of Kato type smoothing estimates, see Lemmas 3.3-5, we follow in spirit [M1] but we also make several simplifications. We recall that [M1] extends to dimension 1 a strengthening of the result by [SW2] due to [GNT]. One of the main ingredients in [GNT] is the endpoint Strichartz estimate, involving spaces $L_t^2 L^{\frac{2D}{D-2}}$, for $D \geq 3$. Here the key point is the L^2 norm in time. The main point in [M1] is the search of a surrogate of the endpoint Strichartz estimate for $D = 1$. In [M1] there are various new interesting smoothing estimates involving the L^2 norm in time. Here we point out that, to show asymptotic stability, the classical smoothing estimates in [K] are sufficient. As a consequence, following our argument, it is possible to prove Theorem 2 [M1], the stability result in [M1], assuming the decay condition $V \in L^{1,1}(\mathbb{R})$ for the potential $V(x)$. Kato [K] smoothing in this paper follows from simple estimates on the resolvent $R_H(z)$. Lemma 3.5 below requires an extension to Birman-Solomjak spaces, proved in [CV2], of a result by Christ & Kiselev [CK], see Lemma 3.1 [SmS].

A substantial part of this paper is dedicated in reproving the main result in [PS]. We first develop some theory of Jost functions for H , along lines very similar to the first part of [DT]. We then follow very closely the treatment of dispersion in low energies for the continuous case in 1D contained in [GSc]. In this way we prove:

Theorem 1.3. *Assume $q \in \ell^{1,1}$ if H has no resonances in 0 and 4, i.e. H satisfies (H2), and $q \in \ell^{1,2}$ if 0 or 4 is a resonance, i.e. H does not satisfy (H2). Then for $P_c(H)$ the projection on the continuous spectrum, we have*

$$\|P_c(H)e^{itH} : \ell^1(\mathbb{Z}) \rightarrow \ell^\infty(\mathbb{Z})\| \leq C\langle t \rangle^{-1/3} \text{ for a fixed } C > 0.$$

Theorem 1.3 (which is restated as Theorem 5.10 and proved in §5) is similar to the main result in [PS]. However here we allow resonances and improve the decay

rates in [PS], which state their result for $q \in \ell^{1,\sigma}(\mathbb{Z})$ with $\sigma > 4$ and only for the non resonant case.

Few weeks after we produced this work, Kevrekidis *et al.* [KPS] produced independently a result similar to our Theorem 1.2 and Theorem 4.1. They assume $q \in \ell^{1,\sigma}$ for $\sigma > 5$ instead of $\sigma = 1$. They prove weaker Strichartz estimates than the ones in Theorem 4.1 since they do not exploit the fact that the upper bound in the dispersive estimate in Theorem 1.3 is not singular for $t = 0$. They do not discuss asymptotic completeness. They reproduce the smoothing argument in [M1]. In doing so they avoid some of the difficulties in [M1] thanks to the fact that energy and ℓ^2 norms coincide in the discrete setting. Nonetheless, this treatment of smoothing, as well as reliance on [PS], is responsible for the more restrictive hypotheses on the decay of $q(n)$ required in [KPS].

We end with some notation. Given an operator A we set $R_A(z) = (A - z)^{-1}$ its resolvent. Given a function $f(\theta)$ for $\theta \in [-\pi, \pi]$, we denote by $\widehat{f}(n)$ or by $f^\wedge(n)$ its n -th Fourier coefficient, writing Fourier series in terms of exponentials. We set $f^\vee(n) = f^\wedge(-n)$. We set $\dot{f} = \partial_\theta f$; $n^\pm = \max(\pm n, 0)$. We set

$$\eta(n) = \sum_{m=n}^{\infty} |q(m)| \text{ and } \gamma(n) = \sum_{m=n}^{\infty} (m-n)|q(m)|.$$

We will denote by $\mathcal{S}(\mathbb{Z})$ the set of functions $f(n)$ rapidly decreasing as $|n| \nearrow \infty$. We will denote by $\mathcal{S}(\mathbb{R} \times \mathbb{Z})$ the set of functions $f(t, n)$ rapidly decreasing as (t, n) diverges along with all the derivatives $\partial_t^a f(t, n)$ for $a \in \mathbb{N}$. Given two Banach spaces X and Y , $B(X, Y)$ will be the space of bounded linear operators defined in X with values in Y . By $\mathbb{Z}_{\geq a}$ we mean the subset of \mathbb{Z} formed by elements $\geq a$. For $x \in \mathbb{R}$ the integer part $[x] \in \mathbb{Z}$ is defined by $[x] \leq x < [x] + 1$.

§2 LINEARIZATION, MODULATION AND SET UP

By standard arguments it is possible to prove:

Lemma 2.1(Global well posedness). *The DNLS (1.2) is globally well posed, in the sense that any initial value problem $u(0, n) = u_0(n)$ with $u_0 \in \ell^2$ admits exactly one solution $u(t) \in C^\infty(\mathbb{R}, \ell^2)$. The correspondence $u_0 \rightarrow u(t)$ defines a continuous map $\ell^2 \rightarrow C^\infty([T_1, T_2], \ell^2)$ for any bounded interval $[T_1, T_2]$.*

By an elementary and standard implicit function theorem argument, which we skip, it is possible to prove the following standard lemma:

Lemma 2.2 (Coordinates near standing waves). *Fix ω_0 close to E_0 . Then there are an $\epsilon_0 > 0$ and a $C_0 > 0$ such that any $u_0 \in \ell^2$ with $\|u_0 - \phi_{\omega_0}\|_{\ell^2} \leq \epsilon \leq \epsilon_0$ can be written in a unique way in the form $u_0 = e^{i\gamma(0)}(\phi_{\omega(0)} + r(0))$ with*

$|\omega_0 - \omega(0)| + |\gamma(0)| + \|r(0)\|_{\ell^2} \leq C_0 \epsilon$ and with $\langle \Re r(0), \phi_{\omega(0)} \rangle = \langle \Im r(0), \partial_\omega \phi_{\omega(0)} \rangle = 0$. The correspondence $u_0 \rightarrow (\gamma(0), \omega(0), r(0))$ is a smooth diffeomorphism.

Consider the initial datum $u_0(n)$ which we will suppose close to ϕ_{ω_0} . For some $T > 0$ and for $0 \leq t \leq T$ the corresponding solution $u(t, n)$ can be written as

$$(2.1) \quad u(t, n) = e^{i\Theta(t)}(\phi_{\omega(t)}(n) + r(t, n)) \text{ where } \Theta(t) = \int_0^t \omega(s)ds + \gamma(t).$$

with $(\gamma(t), \omega(t), r(t))$ with C^∞ dependence in t and such that $\langle \Re r(t), \phi_{\omega(t)} \rangle = \langle \Im r(t), \partial_\omega \phi_{\omega(t)} \rangle = 0$ for $0 \leq t \leq T$. When we plough the ansatz in (1.2) we obtain

$$(2.2) \quad i\partial_t r(t, n) = (Hr)(t, n) + \omega(t)r(t, n) - 4\phi_{\omega(t)}^6(n)r(t, n) - 3\phi_{\omega(t)}^6(n)\bar{r}(t, n) \\ + \dot{\gamma}(t)\phi_{\omega(t)}(n) - i\dot{\omega}(t)\partial_\omega \phi_{\omega(t)}(n) + \dot{\gamma}(t)r(t, n) + N(r(t, n))$$

for $N(r(t, n)) = O(r^2(t, n))$. In particular

$$(2.3) \quad |N(r(t, n))| \leq c_0(|\phi_{\omega}^5(n)r^2(n)| + |r(n)|^7)$$

for a fixed c_0 . The condition $\langle \Re r(t), \phi_{\omega(t)} \rangle = \langle \Im r(t), \partial_\omega \phi_{\omega(t)} \rangle = 0$. yields

$$(2.4) \quad \mathcal{A}(t) \begin{bmatrix} \dot{\omega} \\ \dot{\gamma} \end{bmatrix} = \begin{bmatrix} \langle \Im N, \phi_{\omega(t)} \rangle \\ \langle \Re N, \partial_\omega \phi_{\omega(t)} \rangle \end{bmatrix}$$

$$(2.5) \quad \text{with } \mathcal{A}(t) = \begin{bmatrix} \frac{1}{2}\partial_\omega \|\phi_{\omega}\|_{\ell^2}^2 - \langle \Re r, \partial_\omega \phi_{\omega} \rangle & \langle \Im r, \phi_{\omega} \rangle \\ \langle \Im r, \partial_\omega^2 \phi_{\omega} \rangle & -\frac{1}{2}\partial_\omega \|\phi_{\omega}\|_{\ell^2}^2 - \langle \Re r, \partial_\omega \phi_{\omega} \rangle \end{bmatrix}.$$

In order to prove Theorem 1.2 we follow a scheme introduced by Mizumachi [M1], in particular we will need a number of dispersive estimates on e^{-itH} . Specifically in Theorem 5.10 we prove:

§3 SPACETIME ESTIMATES FOR H

We list a number of linear estimates needed in the stability argument. Given an operator H as in (1.1) we will denote by $P_d(H)$ the spectral projection on the discrete spectrum of H and we will set $P_c(H) = 1 - P_d(H)$. We set $\ell_c^2(H) = P_c(H)\ell^2$. The first result, due to Pelinovsky & Stefanov [PS] but which we strengthen, is Theorem 1.3, see Theorem 5.10 below. Our next step are the Strichartz estimates. Here we follow an idea in [CV1], for much earlier discussion see [GV]. For every $1 \leq p, q \leq \infty$ we introduce the Birman-Solomjak spaces

$$\ell^p(\mathbb{Z}, L_t^q[n, n+1]) \equiv \left\{ f \in L_{loc}^q(\mathbb{R}) \text{ s.t. } \{\|f\|_{L^q[n, n+1]}\}_{n \in \mathbb{Z}} \in \ell^p(\mathbb{Z}) \right\},$$

endowed with the norms

$$\begin{aligned}\|f\|_{\ell^p(\mathbb{Z}, L_t^q[n, n+1])}^p &\equiv \sum_{n \in \mathbb{Z}} \|f\|_{L_t^q[n, n+1]}^p \quad \forall \quad 1 \leq p < \infty \text{ and } 1 \leq q \leq \infty \\ \|f\|_{\ell^\infty(\mathbb{Z}, L_t^q[n, n+1])} &\equiv \sup_{n \in \mathbb{Z}} \|f\|_{L^q[n, n+1]}.\end{aligned}$$

We will say that a pair of numbers (r, p) is admissible if

$$(3.1) \quad 2/r + 1/p = 1/2 \text{ and } (r, p) \in [4, \infty] \times [2, \infty].$$

Then by the standard TT^* argument it is possible to prove from Theorem 1.3 the following result, whose proof we skip, but see Lemma 3.2 in [CV2] and the proof in §9 [CV2]:

Lemma 3.1 (Strichartz estimates). *Under the hypotheses and conclusions of Theorem 1.3 there exists a constant $C = C_H$ such that for every admissible pair (r, p) we have:*

$$\|e^{itH} P_c(H)f\|_{\ell^{\frac{3}{2}r}(\mathbb{Z}, L_t^\infty([n, n+1], \ell^p(\mathbb{Z})))} \leq C\|f\|_{\ell^2(\mathbb{Z})}.$$

Moreover, for any two admissible pairs $(r_1, p_1), (r_2, p_2)$ we have the estimate

$$\begin{aligned}&\left\| \int_0^t e^{i(t-s)H} P_c(H)g(s)ds \right\|_{\ell^{\frac{3}{2}r_1}(\mathbb{Z}, L_t^\infty([n, n+1], \ell^{p_1}(\mathbb{Z})))} \leq \\&\leq C\|g\|_{\ell^{\left(\frac{3}{2}r_2\right)'}(\mathbb{Z}, L_t^1([n, n+1], \ell^{p_2}(\mathbb{Z})))}.\end{aligned}$$

In §5 we prove the following Kato smoothness result:

Lemma 3.2. *Assume that H is generic with $q \in \ell^{1,1}$. For $\tau > 1$ there exists $C = C(\tau)$ such that for all $z \in \mathbb{C} \setminus [0, 4]$*

$$\|R_H(z)P_c(H)\|_{B(\ell^{2,\tau}, \ell^{2,-\tau})} \leq C.$$

The following limits are well defined for any $\lambda \in [0, 4]$ in $C^0([0, 4], B(\ell^{2,\tau}, \ell^{2,-\tau}))$

$$\lim_{\epsilon \rightarrow 0^+} R_H(\lambda \pm i\epsilon) = R_H^\pm(\lambda).$$

For any $u \in \ell^{2,\tau} \cap \ell_c^2(H)$ we have

$$P_c(H)u = \frac{1}{2\pi i} \int_{\mathbb{R}} (R_H^+(\lambda) - R_H^-(\lambda))ud\lambda = \frac{1}{2\pi i} \int_0^4 (R_H^+(\lambda) - R_H^-(\lambda))ud\lambda.$$

The next few lemmas are simplifications of corresponding lemmas in [M1]. First of all we skip the smoothing estimates in Lemma 4 [M1] and we instead consider the following classical consequence of Lemma 3.2, see [K]:

Lemma 3.3. Assume that H is generic with $q \in \ell^{1,1}$. Then for $\tau > 1$ we have:

(a) for any $f \in \mathcal{S}(\mathbb{Z})$ and for $C(\tau)$ the constant of Lemma 3.2

$$\|e^{-itH} P_c(H)f\|_{\ell^2, -\tau L_t^2} \leq 2\sqrt{\pi C(\tau)} \|f\|_{\ell^2};$$

(b) for any $g(t, n) \in \mathcal{S}(\mathbb{R} \times \mathbb{Z})$

$$\left\| \int_{\mathbb{R}} e^{itH} P_c(H)g(t, \cdot) dt \right\|_{\ell^2} \leq 2\sqrt{\pi C(\tau)} \|g\|_{\ell^2, \tau L_t^2}.$$

Proof. (a) implies (b) by duality. So we focus on (a), which is a consequence of §5 [K]. Get $g(t, \nu) \in S(\mathbb{R} \times \mathbb{Z})$ with $g(t) = P_c(H)g(t)$. By the limiting absorption principle in Lemma 3.2

$$\langle e^{-itH} f, g \rangle_{L_t^2 \ell^2} = \frac{1}{2\pi i} \int_0^4 \left\langle (R_H^+(\lambda) - R_H^-(\lambda))f, \bar{g}(\lambda) \right\rangle_{\ell^2} d\lambda.$$

Then from Fubini and Plancherel we have

$$(3.2) \quad |\langle e^{-itH} f, g \rangle_{L_t^2 \ell^2}| \leq (2\pi)^{-\frac{1}{2}} \|(R_H^+(\lambda) - R_H^-(\lambda))f\|_{\ell^2, -\tau L_\lambda^2(0,4)} \|g\|_{\ell^2, \tau L_t^2}.$$

We have

$$\|(R_H^+(\lambda) - R_H^-(\lambda))f\|_{\ell^2, -\tau L_\lambda^2(0,4)} = \lim_{\varepsilon \searrow 0} \|(R_H(\lambda + i\varepsilon) - R_H(\lambda - i\varepsilon))f\|_{\ell^2, -\tau L_\lambda^2(0,4)}.$$

Then we repeat an argument in Lemma 5.5 [K]. For $\Im \mu > 0$ set $K(\mu)$ the positive square root of $(2\pi i)^{-1}[R_H(\mu) - R_H(\overline{\mu})] = \pi^{-1}(\Im \mu)R_H(\overline{\mu})R_H(\mu)$. Then

$$(3.3) \quad \begin{aligned} & \int_0^4 \|(R_H(\lambda + i\varepsilon) - R_H(\lambda - i\varepsilon))f\|_{\ell^2, -\tau}^2 d\lambda = \\ & 4\pi^2 \int_0^4 \|\langle n \rangle^{-\tau} K(\lambda + i\varepsilon) K(\lambda - i\varepsilon) f\|_{\ell^2}^2 d\lambda \leq 8\pi^2 C(\tau) \int_0^4 \|K(\lambda + i\varepsilon) f\|_{\ell^2}^2 d\lambda \\ & \leq 8\pi^2 C(\tau) \|f\|_{\ell^2}^2 \end{aligned}$$

with $C(\tau)$ the constant in Lemma 3.2. By (3.3) we get in (3.2)

$$(3.4) \quad |\langle e^{-itH} f, g \rangle_{L_t^2 \ell^2}| \leq 2\sqrt{\pi C(\tau)} \|f\|_{\ell^2} \|g\|_{\ell^2, \tau L_t^2}.$$

(3.4) yields claim (a).

Lemma 3.4. Assume H generic with $q \in \ell^{1,1}$. Then for any $\tau > 1 \exists C_\tau$ s.t.

$$\left\| \int_0^t e^{-i(t-s)H} P_c(H) g(s, \cdot) ds \right\|_{\ell^{2,-\tau} L_t^2} \leq C_\tau \|g\|_{\ell^{2,\tau} L_t^2}.$$

Proof. By Plancherel and Hölder inequalities and by Lemma 3.2 we have

$$\begin{aligned} & \left\| \int_0^t e^{-i(t-s)H} P_c(H) g(s, \cdot) ds \right\|_{L_t^2 \ell^{2,-\tau}} \leq \\ & \leq \|R_H^+(\lambda) P_c(H) \widehat{\chi}_{[0,+\infty)} *_\lambda \widehat{g}(\lambda, x)\|_{L_\lambda^2 \ell^{2,-\tau}} \leq \\ & \leq \| \|R_H^+(\lambda) P_c(H)\|_{B(\ell^{2,\tau}, \ell^{2,-\tau})} \| \widehat{\chi}_{[0,+\infty)} *_\lambda \widehat{g}(\lambda, x) \|_{\ell^{2,\tau}} \|_{L_\lambda^2} \\ & \leq \sup_{\lambda \in \mathbb{R}} \|R_H^+(\lambda) P_c(H)\|_{B(\ell^{2,\tau}, \ell^{2,-\tau})} \|g\|_{L_t^2 \ell^{2,\tau}} \lesssim \|g\|_{L_t^2 \ell^{2,\tau}}. \end{aligned}$$

Lemma 3.5. Assume H generic with $q \in \ell^{1,1}$. Then for every $\tau > 1 \exists C_\tau$ s. t.

$$\left\| \int_0^t e^{-i(t-s)H} P_c(H) g(s, \cdot) ds \right\|_{L_t^\infty \ell^2 \cap \ell^6(\mathbb{Z}, L_t^\infty([n, n+1], \ell^\infty))} \leq C \|g\|_{L_t^2 \ell^{2,\tau}}.$$

Proof. For $g(t, \nu) \in S(\mathbb{R} \times \mathbb{Z})$ set

$$Tg(t) = \int_0^{+\infty} e^{-i(t-s)H} P_c(H) g(s) ds.$$

Lemma 3.3 (b) implies $f := \int_0^{+\infty} e^{isH} P_c(H) g(s) ds \in \ell^2$. For (r, p) admissible we have

$$\|Tg(t)\|_{\ell^{\frac{3}{2}r}(\mathbb{Z}, L_t^\infty([n, n+1], \ell^p))} \lesssim \|f\|_{\ell^2} \lesssim \|g\|_{L_t^2 \ell^{2,\tau}}.$$

Lemma 3.5 follows from this estimate by an extension of the Christ Kiselev Lemma 3.1 [SmS] to Birman - Solomjak spaces. The proof of this extension is in [CV2]. Consider two Banach spaces and X and Y and $K(s, t)$ continuous function valued in the space $B(X, Y)$. Let

$$T_K f(t) = \int_{-\infty}^{\infty} K(t, s) f(s) ds \text{ and } \tilde{T}_K f(t) = \int_{-\infty}^t K(t, s) f(s) ds.$$

Then we have:

Lemma 3.6. Let $1 \leq p, q, r \leq \infty$ be such that $1 \leq r < \min(p, q) \leq \infty$. Assume that there exists $C > 0$ such that

$$\|T_K f\|_{\ell^q(\mathbb{Z}, L_t^p([n, n+1], Y))} \leq C \|f\|_{L_t^r(X)}.$$

Then

$$\|\tilde{T}_K f\|_{\ell^q(\mathbb{Z}, L_t^p([n, n+1], Y))} \leq C' \|f\|_{L_t^r(X)}$$

where $C' = C'(C, p, q, r) > 0$.

In the case $p = q$ the previous lemma follows from [CK], while the general case is in [CV2].

§4 PROOF OF THEOREM 1.2

Our first goal here is to prove the following:

Theorem 4.1. *Fix $\omega_0 \in]E_0, E_0 + \eta[$ with $\eta > 0$ sufficiently small. Then there exist an $\epsilon_0 > 0$ and a $C > 0$ such that if we pick $\|u_0 - \phi_{\omega_0}\|_{\ell^2} < \epsilon < \epsilon_0$, then the ansatz (2.1) with $\langle \Re r(t), \phi_{\omega(t)} \rangle = \langle \Im r(t), \partial_\omega \phi_{\omega(t)} \rangle = 0$ is valid for all times and we have the following inequalities:*

- (1) $\|(\dot{\omega}, \dot{\gamma})\|_{L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})} \leq C\epsilon,$
- (2) $\|r(t, n)\|_{\ell^{\frac{3}{2}r}(\mathbb{Z}, L_t^\infty([n, n+1], \ell^p))} \leq C\epsilon \text{ for all admissible } (r, p).$

Remark. Notice that (1) implies the existence of γ_\pm and ω_\pm such that

$$\lim_{t \rightarrow \pm\infty} (\omega, \gamma)(t) = (\omega_\pm, \gamma_\pm).$$

By Lemma 2.2 we conclude that (1) implies that

$$|(\omega_0, 0) - (\omega, \gamma)(t)| \leq C(\omega_0, C)\epsilon \text{ for all } t.$$

Proof of Theorem 4.1. There are two equivalent ways to prove results like Theorem 4.1. One way it to prove estimates (1-2) over bounded intervals $[-T, T]$ with constants ϵ_0, C independent of T and then let $T \nearrow \infty$. However one can reach the same result by assuming that the global space-time estimates hold for some large constant C_1 , and then by showing that the estimates hold also for $C_1/2$. By standard arguments, this sort of a priori estimates method yields Theorem 4.1. Set $X_{(r,p)} = \ell^{\frac{3}{2}r}(\mathbb{Z}, L_t^\infty([n, n+1], \ell^p))$. We will then prove:

Lemma 4.2. *Fix $\omega_0 \in]E_0, E_0 + \eta[$ and $\eta > 0$ small. For D with*

$$(1) \quad 1 < D < \eta^{-1},$$

there exists an $\epsilon_0 = \epsilon_0(D)$ with

$$(2) \quad 0 < \epsilon_0 < (\omega_0 - E_0)^7,$$

such that, if for an u_0 with $\|u_0 - \phi_{\omega_0}\|_{\ell^2} < \epsilon < \epsilon_0$ ansatz (2.1) with $\langle \Re r(t), \phi_{\omega(t)} \rangle = \langle \Im r(t), \partial_\omega \phi_{\omega(t)} \rangle = 0$ is valid for all times and if we have

- (3) $\|(\dot{\omega}, \dot{\gamma})\|_{L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})} \leq D\epsilon,$
- (4) $\|r(t, n)\|_{X_{(r,p)} \cap L_t^2 \ell^{2,-2}} \leq D\epsilon \text{ for all admissible } (r, p),$

then for some fixed C_0 and for all admissible (r, p)

$$(5) \quad \|(\dot{\omega}, \dot{\gamma})\|_{L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})} \leq \epsilon,$$

$$(6) \quad \|r(t, n)\|_{X_{(r, p)} \cap L_t^2 \ell^{2, -2}} \leq C_0 \epsilon.$$

It is enough to prove Lemma 4.2 to obtain Theorem 4.1. Set

$$P_d(\omega) = 2 \frac{\langle \Re \cdot, \phi_\omega \rangle}{\partial_\omega \|\phi_\omega\|_{\ell^2}^2} \partial_\omega \phi_\omega + 2i \frac{\langle \Im \cdot, \partial_\omega \phi_\omega \rangle}{\partial_\omega \|\phi_\omega\|_{\ell^2}^2} \phi_\omega \text{ and } P_c(\omega) = 1 - P_d(\omega).$$

Then $P_d(\omega)$ and $P_c(\omega)$ are well defined operators in $\ell^{p, \sigma}$ for all $p \geq 1$ and $\sigma \in \mathbb{R}$. We have $P_d(\omega(t))r(t) = 0$ for all t . By Lemma 1.1 we have

$$\begin{aligned} \partial_\omega \phi_\omega &= \frac{1}{6} (\omega - E_0)^{-\frac{5}{6}} \|\varphi_0\|_{\ell^8}^{-\frac{4}{3}} (\varphi_0 + O(\omega - E_0)) \text{ in any } \ell^{p, \sigma}, \\ \partial_\omega \|\phi_\omega\|_{\ell^2}^2 &= \frac{1}{3} (\omega - E_0)^{-\frac{2}{3}} \|\varphi_0\|_{\ell^8}^{-\frac{8}{3}} (1 + O(\omega - E_0)). \end{aligned}$$

With (1.4) these expansions imply

$$\|P_d(\omega) - P_d(H)\|_{B(\ell^{p, \sigma}, \ell^{p, \sigma})} \leq C_{p, \sigma} |\omega - E_0|.$$

Then (1-3) imply $C_{p, \sigma} |\omega(t) - E_0| \leq 1/2$ for all t . So from $r(t) = P_c(\omega(t))r(t)$ we conclude $\|r(t)\|_{\ell^{p, \sigma}}/\|P_c(H)r(t)\|_{\ell^{p, \sigma}} \in [1/2, 2]$. Hence $\|r\|_{X_{(r, p)}}/\|P_c(H)r\|_{X_{(r, p)}}$ and $\|r\|_{L_t^2 \ell^{2, -2}}/\|P_c(H)r\|_{L_t^2 \ell^{2, -2}}$ are in $[1/2, 2]$. We rewrite (2.2) in the form

$$\begin{aligned} i\partial_t r(t, n) &= (Hr)(t, n) + \omega(t)r(t, n) + \dot{\gamma}(t)r(t, n) + \sum_{j=2}^4 g_j, \\ (4.1) \quad g_2(t, n) &= -4\phi_{\omega(t)}^6(n)r(t, n) - 3\phi_{\omega(t)}^6(n)\bar{r}(t, n), \\ g_3(t, n) &= \dot{\gamma}(t)\phi_{\omega(t)}(n) - i\dot{\omega}(t)\partial_\omega \phi_{\omega(t)}(n), \\ g_4(t, n) &= N(r(t, n)). \end{aligned}$$

Set $w(t) = e^{i\Theta(t)}r(t)$ with $\Theta(t) = \int_0^t \omega(s)ds + \gamma(t)$. Obviously $\|P_c(H)r\|_{X_{(r, p)}} = \|P_c(H)w\|_{X_{(r, p)}}$ $\forall X_{(r, p)}$ and $\|P_c(H)r\|_{L_t^2 \ell^{2, -2}} = \|P_c(H)w\|_{L_t^2 \ell^{2, -2}}$. We have

$$\begin{aligned} (4.2) \quad P_c(H)w(t) &= e^{-iHt} P_c(H)w(0) + \sum_{j=2}^4 w_j(t), \\ w_j(t) &= -i \int_0^t e^{-iH(t-s)} e^{i\Theta(s)} P_c(H)g_j(s) ds. \end{aligned}$$

The following implies Lemma 4.2:

Lemma 4.3. *Assume the hypotheses of Lemma 4.2. Then we have for some fixed C_1 , not dependent on ω_0 or any other parameter, and for all admissible (r, p)*

- $$(7) \quad \|\dot{\omega}\|_{L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})} \leq C_1 D^2 (\omega_0 - E_0)^{5/3} \epsilon^2$$
- $$(8) \quad \|\dot{\gamma}\|_{L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})} \leq C_1 D^2 (\omega_0 - E_0)^{2/3} \epsilon^2$$
- $$(9) \quad \|w_2(t, n)\|_{X_{(r, p)} \cap L_t^2 \ell^{2, -2}} \leq C_1 D (\omega_0 - E_0) \epsilon,$$
- $$(10) \quad \|w_3(t, n)\|_{X_{(r, p)} \cap L_t^2 \ell^{2, -2}} \leq C_1 D^2 (\omega_0 - E_0)^{\frac{5}{6}} \epsilon^2,$$
- $$(11) \quad \|w_4(t, n)\|_{X_{(r, p)} \cap L_t^2 \ell^{2, -2}} \leq C_1 D^2 (\omega_0 - E_0)^{\frac{5}{6}} \epsilon^2.$$

Notice that by Lemmas 3.1-2 we have for fixed constants

$$\|e^{-itH} P_c(H) w(0)\|_{X_{(r, p)} \cap L_t^2 \ell^{2, -2}} \lesssim \|w(0)\|_{\ell^2} \lesssim \epsilon.$$

Then Lemma 4.3 and hypotheses (1)–(2) in Lemma 4.2 imply for some fixed C_H

$$\|P_c(H) w(t)\|_{X_{(r, p)} \cap L_t^2 \ell^{2, -2}} \approx \|w(t)\|_{X_{(r, p)} \cap L_t^2 \ell^{2, -2}} \leq C_H \epsilon.$$

This yields Lemma 4.2.

We focus now on Lemma 4.3. We observe that (2)–(3) yield $\frac{|\omega - E_0|}{|\omega_0 - E_0|} \in [1/2, 3/2]$ by

$$|\omega(t) - \omega_0| \leq \|\dot{\omega}\|_{L^1} + |\omega(0) - \omega_0| \leq (D + C_0) \epsilon < |\omega_0 - E_0|/2,$$

where we used also Lemma 2.2. Now we prove (7–11). We start from (7). By the hypotheses in Lemma 4.2, $\mathcal{A}(t) \approx \partial_\omega \|\phi_\omega\|_{\ell^2}^2 2^{-1} \text{diag}(1, -1) + o(\partial_\omega \|\phi_\omega\|_{\ell^2}^2)$ in $GL(2)$, with \mathcal{A} the matrix in (2.5). Then $\|\mathcal{A}^{-1}(t)\|_{GL(2)} \lesssim (\omega_0 - E_0)^{2/3}$. Thus by (2.3)–(2.4), by (2) and by (4)

$$(12) \quad \begin{aligned} |\dot{\omega}(t)| &\lesssim (\omega_0 - E_0)^{\frac{2}{3}} \|\phi_\omega\|_{\ell^\infty} \left(\|\phi_\omega\|_{\ell^\infty}^5 \|w(t)\|_{\ell^2}^2 + \|w(t)\|_{\ell^2}^7 \right), \\ \|\dot{\omega}\|_{L_t^1} &\lesssim (\omega_0 - E_0)^{\frac{2}{3}} \|\phi_\omega\|_{L_t^\infty \ell^{\infty, 4}} \left(\|\phi_\omega\|_{L_t^\infty \ell^\infty}^5 \|w\|_{L_t^2 \ell^{2, -2}}^2 + \|w\|_{L_t^\infty \ell^2} \|w\|_{L_t^6 \ell^\infty}^6 \right). \end{aligned}$$

So $\|\dot{\omega}\|_{L^1 \cap L^\infty} \lesssim D^2 (\omega_0 - E_0)^{\frac{5}{3}} \epsilon^2$. Similarly $\|\dot{\gamma}\|_{L^1 \cap L^\infty} \lesssim D^2 (\omega_0 - E_0)^{\frac{2}{3}} \epsilon^2$ by

$$(13) \quad \begin{aligned} |\dot{\gamma}(t)| &\lesssim (\omega_0 - E_0)^{\frac{2}{3}} \|\partial_\omega \phi_\omega\|_{\ell^\infty} \left(\|\phi_\omega\|_{\ell^\infty}^5 \|w(t)\|_{\ell^2}^2 + \|w(t)\|_{\ell^2}^7 \right), \\ \|\dot{\gamma}\|_{L_t^1} &\lesssim (\omega_0 - E_0)^{\frac{2}{3}} \|\partial_\omega \phi_\omega\|_{L_t^\infty \ell^{\infty, 4}} \left(\|\phi_\omega\|_{L_t^\infty \ell^\infty}^5 \|w\|_{L_t^2 \ell^{2, -2}}^2 + \|w\|_{L_t^\infty \ell^2} \|w\|_{L_t^6 \ell^\infty}^6 \right). \end{aligned}$$

We have for $j = 2, 3, 4$

$$(14) \quad \|w_j(t)\|_{X_{(r,p)} \cap L_t^2 \ell^{2,-2}} \leq \left\| \int_0^t e^{-iH(t-s)} e^{i\Theta(s)} g_j(s) ds \right\|_{X_{(r,p)} + L_t^2 \ell^{2,-2}}.$$

For $j = 2, 3$ the latter $\lesssim \|g_j\|_{\ell^{2,2} L_t^2}$ by Lemmas 3.4-5. We obtain

$$(4.3) \quad \begin{aligned} \|g_2\|_{\ell^{2,2} L_t^2} &\leq 7 \|\phi_\omega\|_{\ell^\infty,2 L_t^\infty}^6 \|w\|_{\ell^{2,-2} L_t^2} \\ &\lesssim (\omega_0 - E_0) \|w\|_{\ell^{2,-2} L_t^2} \leq D(\omega_0 - E_0) \epsilon \leq \epsilon. \end{aligned}$$

Moreover we get $\|g_3\|_{\ell^{2,2} L_t^2} \leq \|\dot{\omega} \partial_\omega \phi_\omega\|_{\ell^{2,2} L_t^2} + \|\dot{\gamma} \phi_\omega\|_{\ell^{2,2} L_t^2}$. Then by (12)–(13) we have

$$(4.4) \quad \begin{aligned} \|g_3\|_{\ell^{2,2} L_t^2} &\leq \|\dot{\omega}\|_{L_t^2} \|\partial_\omega \phi_\omega\|_{L_t^\infty \ell^{2,2}} + \|\dot{\gamma}\|_{L_t^2} \|\phi_\omega\|_{L_t^\infty \ell^{2,2}} \\ &\lesssim D^2 (\omega_0 - E_0)^{\frac{5}{6}} \epsilon^2 < \epsilon. \end{aligned}$$

Finally we need to bound $\|w_4\|_{X_{(r,p)} \cap L_t^2 \ell^{2,-2}}$. We split $|g_4| \lesssim |\phi_\omega|^5 |w|^2 + |w|^7$. Correspondingly we have $w_4 = w_{4,1} + w_{4,2}$ with, by the above arguments

$$(4.5) \quad \|w_{4,1}\|_{X_{(r,p)} \cap L_t^2 \ell^{2,-2}} \lesssim (\omega_0 - E_0)^{\frac{5}{6}} \|w\|_{L_t^2 \ell^{2,-2}}^2 \leq D^2 (\omega_0 - E_0)^{\frac{5}{6}} \epsilon^2.$$

By Lemma 3.1,

$$(4.6) \quad \begin{aligned} \|w_{4,2}\|_{X_{(r,p)}} &\leq C_0 \|w^7\|_{L_t^1 \ell^2} \leq C_0 \|w\|_{L_t^\infty \ell^2} \|w\|_{L_t^6 \ell^\infty}^6 \\ &\leq C_0 \|w\|_{L_t^\infty \ell^2} \|w\|_{\ell^6(\mathbb{Z}, L_t^\infty[n, n+1]), \ell^\infty}^6 \leq C_0 D^7 \epsilon^7. \end{aligned}$$

By (b) Lemma 3.3 and by the argument in (4.6)

$$(4.7) \quad \begin{aligned} \|w_{4,2}\|_{L_t^2 \ell^{2,-2}} &\leq \int_0^\infty \|e^{-i(t-s)H} P_c(H) O(w^7)(s)\|_{L_t^2 \ell^{2,-2}} ds \\ &\leq C_H \int_0^\infty \|O(w^7)(s)\|_{\ell^2} ds \lesssim D^7 \epsilon^7. \end{aligned}$$

Then (4.5)–(4.7) yield estimate (11) in Lemma 4.3 by hypothesis (2) in Lemma 4.2. This completes the proof of Lemma 4.3.

The following standard lemma yields the asymptotic flatness of $r(t)$.

Lemma 4.4. *Consider the $r(t, n)$ in Theorem 4.1. Then there exist $r_\pm \in \ell^2$ such that $\|r_\pm\|_{\ell^2} \leq C\epsilon$ for fixed $C = C(\omega_0)$ and*

$$\lim_{t \rightarrow \pm\infty} \|r(t) - e^{-it\Delta} r_\pm\|_{\ell^2} = 0.$$

Proof. We first write (4.2) as

$$\begin{aligned} e^{iHt} P_c(H) w(t) &= P_c(H) w(0) + \sum_{j=2}^4 e^{iHt} w_j(t), \\ e^{iHt} w_j(t) &= -i \int_0^t e^{iHs} e^{i\Theta(s)} P_c(H)(s) g_j(s) ds. \end{aligned}$$

Then we observe that for $t_2 > t_1$ by (4.3-7)

$$\|e^{iHt_2} w_j(t_2) - e^{iHt_1} w_j(t_1)\|_{\ell^2} \leq \|g_j\|_{X_{(r,p)}(t_1, t_2) + \ell^{2,2}(t_1, t_2)} \rightarrow 0 \text{ for } t_1 \rightarrow \infty.$$

This implies that the following limits exist

$$(1) \quad \lim_{t \rightarrow +\infty} e^{iHt} P_c(H) w(t) = \lim_{t \rightarrow +\infty} e^{iHt} w(t) = w_+,$$

with the first equality due to $w(t) \in C^1(\mathbb{R}, \ell^2) \cap \ell^{\frac{3}{2}r}(\mathbb{Z}, L_t^\infty([n, n+1], \ell^p(\mathbb{Z})))$ for any admissible pair (r, p) , so that $\lim_{|t| \rightarrow \infty} P_d(H) w(t) = 0$. Recall that $u(t) = e^{i\Theta(t)} \phi_{\omega(t)} + w(t)$. Then Theorem 4.1 and (1) imply

$$\lim_{t \rightarrow \infty} \|u(t) - e^{i\Theta(t)} \phi_{\omega_+} - e^{-itH} w_+\|_{\ell^2} = 0.$$

By Pearson's Theorem, see Theorem XI.7[RS], the following two limits exist in ℓ^2 , for $w \in \ell_c^2(H)$ and $u \in \ell^2$:

$$Wu = \lim_{t \rightarrow +\infty} e^{itH} e^{it\Delta} u, \quad Zw = \lim_{t \rightarrow +\infty} e^{-it\Delta} e^{-itH} w.$$

This follows from the fact that $H + \Delta = q$ with the operator $u(n) \rightarrow q(n)u(n)$ in the trace class because of $q \in \ell^1$. For $u_+ = Zw_+$ we have the following, which yields Lemma 4.4:

$$\lim_{t \rightarrow \infty} e^{-itH} w_+ = \lim_{t \rightarrow \infty} e^{it\Delta} u_+ \text{ in } \ell^2.$$

§5 DISPERSIVE THEORY FOR H

We recall basic facts concerning the resolvent of the difference Laplace operator. First, for $g(\theta) \in L^2(-\pi, \pi)$ and for $u(n) = \widehat{g}(n)$ we have

$$-(\Delta u)(n) = 2 [(1 - \cos \theta) g(\theta)]^\wedge(n).$$

The resolvent $R_{-\Delta}(z)$ for $z \in \mathbb{C} \setminus [0, 4]$ has kernel

$$R(m, n, z) = \frac{-i}{2 \sin \theta} e^{-i\theta|m-n|}, \quad m, n \in \mathbb{Z},$$

with θ the unique solution to $2(1 - \cos \theta) = z$ in

$$D = \{\theta : -\pi \leq \Re \theta \leq \pi, \Im \theta < 0\}.$$

For all the above see [KKK]. Then $(-\Delta - z)\psi = f, f \in \ell^2(\mathbb{Z})$ has solution

$$\psi_n = (R_{-\Delta}(z)f)_n = \frac{-i}{2\sin \theta} \sum_{m \in \mathbb{Z}} e^{-i\theta|n-m|} f_m.$$

We consider now discrete Jost functions. For z and θ as above we look for functions $f_{\pm}(n, \theta)$ with

$$(5.1) \quad Hf_{\pm}(n, \theta) = zf_{\pm}(n, \theta) \text{ with } \lim_{n \rightarrow \pm\infty} [f_{\pm}(n, \theta) - e^{\mp in\theta}] = 0.$$

Therefore the Green representation of the solutions is

$$(5.2) \quad f_{\pm}(n, \theta) = e^{\mp in\theta} - \sum_{m=n}^{\pm\infty} \frac{\sin(\theta(n-m))}{\sin \theta} q(m) f_{\pm}(m, \theta).$$

Let m_{\pm} be defined by $f_{\pm}(n, \theta) = e^{\mp in\theta} m_{\pm}(n, \theta)$. Then we have

$$(5.3) \quad m_{\pm}(n, \theta) = 1 + \sum_{\nu=n}^{\pm\infty} \frac{1 - e^{2i(n-\nu)\theta}}{2i \sin \theta} q(\nu) m_{\pm}(\nu, \theta).$$

Following standard arguments, see Lemma 1 [DT], we have:

Lemma 5.1. *For $\theta \in \overline{\mathbb{C}_-}$, (5.2) has for any choice of sign a unique solution satisfying the estimates listed below. These solutions solve $Hu = zu$ with the asymptotic property $f_{\pm}(n, \theta) \approx e^{\mp in\theta} + o(e^{\mp in\theta})$ for $n \rightarrow \pm\infty$. For $q \in \ell^{1,\sigma}$, $\sigma \in [1, 2]$, there is a $C = C(q)$ such that $\forall n \in \mathbb{N}$ we have,*

$$(1) \quad |m_{\pm}(n, \theta) - 1| \leq C \langle n^{\pm} \rangle^{-\sigma} |\sin \theta|^{-1} e^{\frac{C}{|\sin \theta|}}, \quad \theta \notin \pi\mathbb{Z},$$

$$(2) \quad |m_{\pm}(n, \theta) - 1| \leq C \langle n^{\pm} \rangle^{-(\sigma-1)} \langle \sin \theta \rangle^{-1} (1 + n^{\mp}).$$

$m_{\pm}(n, \theta)$ are for any n analytic for $\theta \in \mathbb{C}_-$, they satisfy $m_{\pm}(n, \theta) = m_{\pm}(n, \theta + 2\pi)$, and extend into continuous functions in $\overline{\mathbb{C}_-}$.

If $\sigma = 2$, there is a $C = C(q)$ such that

$$(3) \quad |\dot{m}_{+}(n, \theta)| \leq C \langle n^- \rangle^2,$$

$$(4) \quad |\dot{m}_{-}(n, \theta)| \leq C \langle n^+ \rangle^2,$$

where $\dot{m}_\pm(n, \theta)$ are for any n analytic for $\theta \in \mathbb{C}_-$, and extend into continuous functions in $\overline{\mathbb{C}_-}$.

Proof. It is not restrictive to consider $m_+(n, \theta)$. We set $m(n, \theta) = m_+(n, \theta)$ in the rest of this lemma. For $D(n, \theta) := \frac{1-e^{2in\theta}}{2i\sin\theta}$ we have

$$(5.4) \quad m(n, \theta) = 1 + \sum_{\nu=n}^{\infty} D(n-\nu, \theta)q(\nu)m(\nu, \theta).$$

We search for a solution

$$(5) \quad m(n, \theta) = 1 + \sum_{\ell=1}^{\infty} g_\ell(n, \theta),$$

defined recursively by

$$g_\ell(n, \theta) = \sum_{n \leq n_1 \leq \dots \leq n_\ell} D(n - n_1, \theta) \cdots D(n_{\ell-1} - n_\ell, \theta) q(n_1) \cdots q(n_\ell).$$

Notice that the $g_\ell(n, \theta)$ are 2π periodic in θ . Since this is true also for the estimates, we can assume below that $\theta \in \overline{D}$. By $|D(\mu, \theta)| \leq 1/|\sin\theta|$ for $\mu \leq 0$ we get

$$|g_\ell(n, \theta)| \leq \frac{1}{|\sin\theta|^\ell} \sum_{n \leq n_1 \leq \dots \leq n_\ell} |q(n_1) \cdots q(n_\ell)| = \frac{(\sum_{m=n}^{\infty} |q(m)|)^\ell}{\ell! |\sin\theta|^\ell}.$$

Therefore we get the following which yields (1)

$$|m(n, \theta) - 1| \leq \frac{1}{|\sin\theta|} \eta(n) e^{\frac{\eta(n)}{|\sin\theta|}}.$$

We consider now inequality (2). It is enough to assume $\theta \in \overline{D}$ is close either to 0 or to $\pm\pi$, since otherwise estimate (1) is stronger than (2). Then $|D(n, \theta)| \leq C_0|n|$ for a fixed C_0 . Then we get

$$\begin{aligned} |g_\ell(n, \theta)| &\leq C_0^\ell \sum_{n \leq n_1 \leq \dots \leq n_\ell} (n_1 - n) \cdots (n_\ell - n_{\ell-1}) |q(n_1) \cdots q(n_\ell)| = \\ &= C_0^\ell \frac{(\sum_{m=n}^{\infty} (m-n) |q(m)|)^\ell}{\ell!}, \end{aligned}$$

which yields

$$|m(n, \theta) - 1| \leq C_0 \gamma(n) e^{C_0 \gamma(n)}.$$

Notice that the bound increases exponentially if $n \rightarrow -\infty$. The following chains of inequalities are fulfilled

$$\begin{aligned} |m(n, \theta)| &\leq 1 + \sum_{\nu=n}^{\infty} (\nu - n) |q(\nu)| |m(\nu, \theta)| \leq \\ &\leq 1 + \sum_{\nu=n}^{\infty} \nu |q(\nu)| |m(\nu, \theta)| - n \sum_{\nu=n}^{\infty} |q(\nu)| |m(\nu, \theta)| \\ &\leq 1 + \sum_{\nu=0}^{\infty} \nu |q(\nu)| |m(\nu, \theta)| - n \sum_{\nu=n}^{\infty} |q(\nu)| |m(\nu, \theta)|. \end{aligned}$$

Furthermore we have also

$$1 + \sum_{\nu=0}^{\infty} \nu |q(\nu)| |m(\nu, \theta)| \leq 1 + \gamma(0) e^{\gamma(0)} \sum_{\nu=0}^{\infty} \nu |q(\nu)| =: K < \infty.$$

If we set

$$M(n, \theta) := \frac{m(n, \theta)}{K(1 + |n|)},$$

we get the inequality

$$|M(n, \theta)| \leq 1 + \sum_{\nu=n}^{\infty} (1 + |\nu|) |q(\nu)| |M(\nu, \theta)|,$$

which can be solved by an iteration argument as in the previous case obtaining

$$|M(n, \theta)| \leq e^{\sum_{\nu=n}^{\infty} (1 + |\nu|) |q(\nu)| |M(\nu, \theta)|} \leq K_1 < \infty,$$

and this gives

$$|m(n, \theta)| \leq K_2 (1 + |n|).$$

Thus we get (2) by

$$\begin{aligned} |m(n, \theta) - 1| &\leq \sum_{\nu=0}^{\infty} \nu |q(\nu)| |m(\nu, \theta)| - n \sum_{\nu=n}^{\infty} |q(\nu)| |m(\nu, \theta)| \leq \\ &\leq \gamma(0) e^{\gamma(0)} \sum_{\nu=0}^{\infty} \nu |q(\nu)| - K_2 n \sum_{\nu=n}^{\infty} (1 + |\nu|) |q(\nu)| \leq \\ &\leq K_3 (1 + \max(-n, 0)) \sum_{\nu=n}^{\infty} (1 + |\nu|) |q(\nu)|. \end{aligned}$$

Notice that the above arguments yields also the uniqueness of $m(n, \theta)$. The $m(n, \theta)$ are defined for all θ with $\Im\theta \leq 0$. They satisfy $m(n, \theta) = m(n, \theta + 2\pi)$ and are analytic for $\Im\theta < 0$. The last two properties follow from the fact that the $g(n, \theta)$ satisfy these properties and that series (5) converges uniformly for $\Im\theta \leq 0$.

We now prove (3) (the proof of (4) is similar). Better estimates than (3) can be obtained using (1), the analyticity of $m(n, \theta)$ for $\Im\theta \leq 0$ and the Cauchy integral formula. So it is enough to assume that $\theta \in \overline{D}$ is close to the interval $[-\pi, \pi]$. Differentiating (5.4) we get

$$\dot{m}(n, \theta) = \sum_{\nu=n}^{\infty} D(n - \nu, \theta) q(\nu) \dot{m}(\nu, \theta) + \sum_{\nu=n}^{\infty} \dot{D}(n - \nu, \theta) q(m) m(\nu, \theta).$$

We consider the representations

$$\begin{aligned} D(n - \nu, \theta) &= \frac{\theta}{\sin \theta} \int_{n-\nu}^0 e^{2i\theta t} dt \text{ for } \Re\theta \in [-\pi/2, \pi/2], \\ D(n - \nu, \theta) &= \frac{\theta \mp \pi}{\sin \theta} \int_{n-\nu}^0 e^{2i(\theta \mp \pi)t} dt \text{ for } \Re\theta \in [\pi/2, \pi] \text{ (resp. } \Re\theta \in [-\pi, \pi/2]). \end{aligned}$$

Then

$$(5.5) \quad \dot{D}(n - \nu, \theta) = \partial_{\theta} \left(\frac{\theta}{\sin \theta} \right) \int_{n-\nu}^0 e^{2i\theta t} dt + 2i \frac{\theta}{\sin \theta} \int_{n-\nu}^0 t e^{2i\theta t} dt,$$

$$(5.6) \quad \dot{D}(n - \nu, \theta) = \partial_{\theta} \left(\frac{\theta \mp \pi}{\sin \theta} \right) \int_{n-\nu}^0 e^{2i(\theta \mp \pi)t} dt + 2i \frac{\theta \mp \pi}{\sin \theta} \int_{n-\nu}^0 t e^{2i(\theta \mp \pi)t} dt.$$

By $n - \nu \leq 0$, (5.5) implies for θ close to $[-\pi, \pi]$

$$|\theta \dot{D}(n - \nu, \theta)| \leq C|n - \nu|, \quad \Re\theta \in [-\pi/2, \pi/2].$$

Similarly by (5.6) we obtain

$$|(\theta - \pi) \dot{D}(n - \nu, \theta)| \leq C|n - \nu|, \quad \Re\theta \in [\pi/2, \pi],$$

and

$$|(\theta + \pi) \dot{D}(n - \nu, \theta)| \leq C|n - \nu|, \quad \Re\theta \in [-\pi, -\pi/2].$$

Furthermore we have the inequality

$$|\dot{D}(n - \nu, \theta)| = \left| \partial_\theta \left(1 - \frac{e^{2i(n-\nu)\theta}}{2i \sin \theta} \right) \right| \leq C(n - \nu)^2,$$

and so

$$\left| \sum_{\nu=n}^{\infty} \dot{D}(n - \nu, \theta) q(\nu) m(\nu, \theta) \right| \leq C \sum_{\nu=n}^{\infty} (n - \nu)^2 |q(\nu) m(\nu, \theta)|.$$

Suppose $n < 0$. Then from the fact that $q \in \ell^{1,2}$

$$\begin{aligned} \sum_{\nu=n}^{\infty} \nu^2 |q(\nu) m(\nu, \theta)| &= \sum_{\nu=n}^0 \nu^2 |q(\nu) m(\nu, \theta)| + \sum_{\nu=0}^{\infty} \nu^2 |q(\nu) m(\nu, \theta)| \\ &\leq \sum_{\nu=n}^0 n^2 |q(\nu) m(\nu, \theta)| + \sum_{\nu=0}^{\infty} \nu^2 |q(\nu) m(\nu, \theta)| \leq K(1 + n^2) \end{aligned}$$

where we used $|m(\nu, \theta)| \leq C \langle \nu^- \rangle$. For $n \geq 0$

$$\sum_{\nu=n}^{\infty} \nu^2 |q(\nu) m(\nu, \theta)| \leq K \sum_{\nu=0}^{\infty} \nu^2 |q(\nu)|.$$

Hence, for all n we obtain

$$\sum_{\nu=n}^{\infty} \nu^2 |q(\nu) m(\nu, \theta)| \leq K \langle n^- \rangle^2.$$

For $n \geq 0$

$$\sum_{\nu=n}^{\infty} (n - \nu)^2 |q(\nu) m(\nu, \theta)| \leq \sum_{\nu=0}^{\infty} \nu^2 |q(\nu) m(\nu, \theta)| \leq K.$$

For $n < 0$ we obtain the chain of inequalities

$$\begin{aligned} \sum_{\nu=n}^{\infty} (n - \nu)^2 |q(\nu) m(\nu, \theta)| &\leq 2 \sum_{\nu=n}^{\infty} \nu^2 |q(\nu) m(\nu, \theta)| + 2n^2 \sum_{\nu=n}^{\infty} |q(\nu) m(\nu, \theta)| \\ &\leq K(1 + \nu^2). \end{aligned}$$

So

$$|\dot{m}(n, \theta)| \leq \sum_{\nu=n}^{\infty} (\nu - n) |q(\nu) \dot{m}(\nu, \theta)| + K_2 \langle n^- \rangle^2,$$

and iterating

$$|\dot{m}(n, \theta)| \leq K_2 \langle n^- \rangle^2 e^{\gamma(n)}.$$

We get, for any $n \in \mathbb{N}$,

$$|\dot{m}(n, \theta)| \leq K_2 \langle n^- \rangle^2 + \sum_{\nu=n}^{\infty} m|q(m)\dot{m}(\nu, \theta)| - n \sum_{\nu=n}^{\infty} |q(\nu)\dot{m}(\nu, \theta)|.$$

The right hand side is smaller than

$$\gamma(0)e^{\gamma(0)} \sum_{\nu=0}^{\infty} m|q(\nu)| - n \sum_{\nu=n}^{\infty} |q(\nu)\dot{m}(\nu, \theta)|,$$

which can be bounded by

$$K_3 \langle n^- \rangle^2 (1 + \sum_{\nu=n}^{\infty} \langle \nu \rangle^2 |q(\nu)|).$$

Following the same line of the proof of (1)-(2), by an iteration argument we get the desired estimate and complete the proof of the Lemma. Analyticity of $\dot{m}(n, \theta)$ in the interior of D and continuity in \overline{D} can be proved as the similar statement for $m(n, \theta)$.

For any fixed n Lemma 5.1 implies the Fourier expansion

$$(5.7) \quad m_{\pm}(n, \theta) = 1 + \sum_{\nu=1}^{\infty} B_{\pm}(n, \nu) e^{-i\nu\theta}.$$

We consider the following:

Lemma 5.2. *For $q \in \ell^{1,1}$ we have*

$$\sup_{n \geq 0} \|B_+(n, \nu)\|_{\ell^1_\nu} < \infty \text{ and } \sup_{n \leq 0} \|B_-(n, \nu)\|_{\ell^1_\nu} < \infty.$$

Proof. It is not restrictive to consider the + case only. We drop the + subscript. By substituting $e^{-in\theta}m(n, \theta) = f_+(n, \theta)$ in (5.1) and using $z = 2 - 2\cos(\theta)$ we obtain

$$e^{-i\theta} (m(n+1, \theta) - m(n, \theta)) + e^{i\theta} (m(n-1, \theta) - m(n, \theta)) = q(n)m(n, \theta).$$

Substituting the Fourier expansion (5.7) we obtain

$$\begin{aligned} & \sum_{\nu=1}^{\infty} (B(n+1, \nu-1) - B(n, \nu-1)) e^{-i\nu\theta} + \\ & \sum_{\nu=0}^{\infty} (B(n-1, \nu+1) - B(n, \nu+1)) e^{-i\nu\theta} = q(n) + q(n) \sum_{\nu=1}^{\infty} B(n, \nu) e^{-i\nu\theta}. \end{aligned}$$

We have $B(n-1, 1) - B(n, 1) = q(n)$ and for $\nu > 0$

$$B(n-1, \nu+1) - B(n, \nu+1) = q(n)B(n, \nu) + B(n, \nu-1) - B(n+1, \nu-1).$$

By induction on ν

$$\begin{aligned} B(n, 2\nu) - B(n+1, 2\nu) &= \sum_{j=1}^{\nu} q(n+j)B(n+j, 2(\nu-j)+1) \\ B(n, 2\nu-1) - B(n+1, 2\nu-1) &= q(n+\nu) + \sum_{j=1}^{\nu} q(n+j)B(n+j, 2(\nu-j)). \end{aligned}$$

So

$$\begin{aligned} B(n, 2\nu) &= \sum_{l=n}^{\infty} \sum_{j=1}^{\nu} q(l+j)B(l+j, 2(\nu-j)+1) \\ B(n, 2\nu-1) &= \sum_{l=n+\nu}^{\infty} q(l) + \sum_{l=n}^{\infty} \sum_{j=1}^{\nu} q(l+j)B(l+j, 2(\nu-j)). \end{aligned}$$

which after a change of variables, and setting $B_+(n, 0) = 0$, we write as

$$(1) \quad \begin{aligned} B(n, 2\nu) &= \sum_{l=0}^{\nu-1} \sum_{j=n+\nu-l}^{\infty} q(j)B(j, 2l+1) \\ B(n, 2\nu-1) &= \sum_{l=n+\nu}^{\infty} q(l) + \sum_{l=0}^{\nu-1} \sum_{j=n+\nu-l}^{\infty} q(j)B(j, 2l). \end{aligned}$$

By Lemma 5.1 (1) we know that $B \in \ell^\infty(\mathbb{Z}_{\geq 0}^2)$. We show now that (1) admits just one solution in this space, which satisfies the bounds in the statement. We consider

$\mathcal{B}(n, \nu) = \sum_{m=0}^{\infty} K_m(n, \nu)$ with

$$(2) \quad \begin{aligned} K_0(n, 2\nu - 1) &= \sum_{l=n+\nu}^{\infty} q(l) \text{ and } K_0(n, 2\nu) = 0 \\ K_m(n, 2\nu) &= \sum_{l=0}^{\nu-1} \sum_{j=n+\nu-l}^{\infty} q(j) K_{m-1}(j, 2l+1) \\ K_m(n, 2\nu - 1) &= \sum_{l=0}^{\nu-1} \sum_{j=n+\nu-l}^{\infty} q(j) K_{m-1}(j, 2l) \end{aligned}$$

By the same inductive argument of p.139 [DT] one can prove

$$(3) \quad |K_m(n, \nu)| \leq \frac{\gamma^m(n)}{m!} \eta(n + [\nu/2]).$$

Indeed (3) is true for $m = 0$. Assume (3) true for m . Then we can write

$$\begin{aligned} |K_{m+1}(n, \nu)| &\leq \sum_{l=0}^{[\nu/2]} \sum_{j=n+[\nu/2]-l}^{\infty} |q(j)| \frac{\gamma^m(j)}{m!} \eta(j + [l/2]) \leq \\ &\leq \eta(n + [\nu/2]) \sum_{l=0}^{[\nu/2]} \sum_{j=n+[\nu/2]-l}^{\infty} |q(j)| \frac{\gamma^m(j)}{m!} \leq \\ &\leq \eta(n + [\nu/2]) \left(\sum_{l=0}^{[\nu/2]} \sum_{j=n+[\nu/2]-l}^{n+[\nu/2]} |q(j)| \frac{\gamma^m(j)}{m!} + \sum_{j=n+[\nu/2]}^{\infty} |q(j)| \frac{\gamma^m(j)}{m!} [\nu/2] \right) \\ &\leq \eta(n + [\nu/2]) \left(\sum_{j=n}^{n+[\nu/2]} |q(j)| \frac{\gamma^m(j)}{m!} (j - n) + \sum_{j=n+\nu}^{\infty} |q(j)| \frac{\gamma^m(j)}{m!} j \right) \leq \\ &\leq \eta(n + [\nu/2]) \sum_{j=n+1}^{\infty} |q(j)| \frac{\gamma^m(j)}{m!} j = \frac{\gamma^{m+1}(n)}{(m+1)!} \eta(n + [\nu/2]). \end{aligned}$$

Thus we have $|\mathcal{B}(n, \nu)| \leq e^{\gamma(n)} \eta(n + [\nu/2])$. Therefore

$$\|\mathcal{B}(n, \nu)\|_{\ell_{\nu}^{\infty}} \leq e^{\gamma(n)} \eta(n) \text{ and } \|\mathcal{B}(n, \nu)\|_{\ell_{\nu}^1} \leq e^{\gamma(n)} \sum_{\nu=0}^{\infty} \eta(n + \nu) \leq e^{\gamma(n)} \gamma(n).$$

Hence $\mathcal{B}(n, \nu)$ satisfies bounds as in the statement. $U(n, \nu) := B(n, \nu) - \mathcal{B}(n, \nu)$ satisfies the equation

$$U(n, 2\nu) = \sum_{l=0}^{\nu-1} \sum_{j=n+\nu-l}^{\infty} q(j) U(j, 2l+1)$$

$$U(n, 2\nu-1) = \sum_{l=0}^{\nu-1} \sum_{j=n+\nu-l}^{\infty} q(j) U(j, 2l).$$

Iterating the above procedure we conclude $U(n, \nu) = 0$.

Given two functions $u(n)$ and $v(n)$ we denote by $[u, v](n) = u(n+1)v(n) - u(n)v(n+1)$ the Wronskian of the pair (u, v) . If u and v are solutions of $Hw = zw$ then $[u, v]$ is constant. Since the equations $Hu = \lambda u$ cannot have all solutions bounded near $+\infty$ we have the following:

Lemma 5.3. *Let $q \in \ell^{1,1}$. Then we have:*

- (1) *If for a $\theta_0 \in \{0, \pi, -\pi\}$ we have $f_+(n, \theta_0) \in \ell^\infty$, then for $W(\theta) := [f_+(\theta), f_-(\theta)]$ we have $W(\theta_0) = 0$. We will call generic an H such that $W(\theta_0) \neq 0$ for all $\theta_0 \in \{0, \pi, -\pi\}$.*
- (2) *No element $\lambda \in [0, 4]$ can be an eigenvalue of H in ℓ^2 .*
- (3) *If $\lambda \in \mathbb{R}$ is an eigenvalue of H , then $\dim(H - \lambda) = 1$.*

Proof. If $q \in \ell^{1,2}$ by Lemma 5.1 we have

$$(3) \quad \lim_{n \rightarrow \pm\infty} \|1 - m_\pm(n, \theta)\|_{L_\theta^\infty} \rightarrow 0.$$

By the fact that the $m_\pm(n, \theta)$ depend continuously on $q \in \ell^{1,1}$, which can be proved using the arguments in Lemma 5.1, (3) is valid also for $q \in \ell^{1,1}$. This and the continuity in θ implies that for both signs $m_\pm(n, \theta) \not\equiv 0$ as a function of n for any fixed θ . If any of the three claims (1-3) is wrong, then for some $\lambda \in \mathbb{R}$ all the solutions of

$$(4) \quad (H - \lambda)u = 0$$

are in ℓ^∞ . Let now $u_1(n)$ and $u_2(n)$ be a fundamental set of such solutions. Consider now the equation $(-\Delta - \lambda)U = 0$ which we rewrite as $(H - \lambda)U = qU$. Then solutions $U \in \ell^\infty([N, \infty))$ can be written for $q \in \ell^1$ as

$$(5) \quad U(n) = u(n) - \sum_{j=n}^{\infty} \frac{u_1(n)u_2(j) - u_2(n)u_1(j)}{[u_1, u_2]} q(j) U(j)$$

with $u(n)$ a solution of (4). But for $q \in \ell^1$ and N large, (5) establishes an isomorphism inside $\ell^\infty([N, \infty))$ between solutions of (4), which form a 2 dimensional space, and of $(-\Delta - \lambda)U = 0$, which form a 1 dimensional space. Obviously this is absurd. Therefore it is not possible for all solutions of (4) to be in ℓ^∞ .

Next we introduce transmission and reflection coefficients.

Lemma 5.4. Let $q \in \ell^{1,1}$. For $\theta \in [-\pi, \pi]$ we have $\overline{f_{\pm}(n, \theta)} = f_{\pm}(n, -\theta)$ and for $\theta \neq 0, \pm\pi$ we have

$$(1) \quad f_{\mp}(n, \theta) = \frac{1}{T(\theta)} \overline{f_{\pm}(n, \theta)} + \frac{R_{\pm}(\theta)}{T(\theta)} f_{\pm}(n, \theta)$$

where $T(\theta)$ and $R_{\pm}(\theta)$ are defined by (1) and satisfy:

$$(2) \quad [\overline{f_{\pm}(\theta)}, f_{\pm}(\theta)] = \pm 2i \sin \theta,$$

$$(3) \quad T(\theta) = \frac{\pm 2i \sin \theta}{[f_{\mp}(\theta), f_{\pm}(\theta)]}, \quad R_{\pm}(\theta) = -\frac{[f_{\mp}(\theta), \overline{f_{\pm}}(\theta)]}{[f_{\mp}(\theta), f_{\pm}(\theta)]},$$

$$(4) \quad \overline{T(\theta)} = T(-\theta), \quad \overline{R_{\pm}(\theta)} = R_{\pm}(-\theta),$$

$$(5) \quad |T(\theta)|^2 + |R_{\pm}(\theta)|^2 = 1, \quad T(\theta) \overline{R_{\pm}(\theta)} + R_{\mp}(\theta) \overline{T(\theta)} = 0.$$

Proof. $\overline{f_{\pm}(n, \theta)} = f_{\pm}(n, -\theta)$ follows by the fact that $q(n)$ has real entries and by uniqueness in Lemma 5.1. The pair in the right hand side of (1) is linearly independent, so (1) follows from the properties of second order homogeneous linear difference equations. (2-4) follow applying Wronskians to (1). Iterating (1) twice we get (5). Indeed, for example

$$\begin{aligned} f_- &= \frac{1}{T} \overline{f_+} + \frac{R_+}{T} f_+ = \frac{1}{T} \left(\frac{1}{\bar{T}} f_- + \frac{\overline{R_-}}{T} f_- \right) + \frac{R_+}{T} \left(\frac{1}{T} \overline{f_-} + \frac{R_-}{T} f_- \right) \\ &= \left(\frac{1}{|T|^2} + \frac{R_+ R_-}{T^2} \right) f_- + \left(\frac{\overline{R_-}}{|T|^2} + \frac{R_+}{T} \right) \overline{f_-}. \end{aligned}$$

This yields $R_- \overline{T} + \overline{R_+} T = 0$ and $\frac{1}{|T|^2} + \frac{R_+ R_-}{T^2} = 1$. Substituting $R_- = -\frac{\overline{R_+} T}{T}$ we get $|T|^2 + |R_+|^2 = 1$. Similarly one gets $|T|^2 + |R_-|^2 = 1$.

Lemma 5.5. Let $W(\theta) := [f_+(\theta), f_-(\theta)]$ and $W_1(\theta) := [f_+(\theta), \overline{f}_-(\theta)]$.

- (1) For $\theta \in [-\pi, \pi] \setminus \{0, \pm\pi\}$ we have $W(\theta) \neq 0$. We have $|W(\theta)| \geq 2|\sin \theta|$ for all $\theta \in [-\pi, \pi]$ and in the generic case $|W(\theta)| > 0$.
- (2) For $j = 0, 1$ and $q \in \ell^{1,1+j}$ then $W(\theta)$ and $W_1(\theta)$ are in $C^j[-\pi, \pi]$.
- (3) If $q \in \ell^{1,2}$ and $W(\theta_0) = 0$ for a $\theta_0 \in \{0, \pm\pi\}$, then $\dot{W}(\theta_0) \neq 0$. In particular if $q \in \ell^{1,2}$, then $T(\theta) = -2i \sin \theta / W(\theta)$ can be extended continuously in $[-\pi, \pi]$ with $T(-\pi) = T(\pi)$.

Proof. (1) follows immediately from $T(\theta) = -2i \sin \theta / W(\theta)$ and $|T(\theta)| \leq 1$ and the definition of H generic in Lemma 5.3. (2) follows from Lemma 5.1. (3) follows from Lemma 5.1 and (1).

We need of the following:

Lemma 5.6. *Let $0 > \Im\theta$ and $q \in \ell^{1,1}$ with $z = 2(1 - \cos\theta)$ not an eigenvalue of H . Then the resolvent $R_H(z)$ has kernel $R_H(n, m, z)$ such that $R_H(n, m, z) = K(n, m)$ where we define*

$$K(n, m) = -\frac{f_-(n, \theta)f_+(m, \theta)}{[f_+(\theta), f_-(\theta)]} \text{ for } n < m$$

$$K(n, m) = -\frac{f_+(n, \theta)f_-(m, \theta)}{[f_+(\theta), f_-(\theta)]} \text{ for } n \geq m.$$

Proof. First of all, if $z \notin [0, 4]$ is not an eigenvalue of H we have $[f_+(\theta), f_-(\theta)] \neq 0$ and $K(n, m)$ is well defined. Indeed if $[f_+(\theta), f_-(\theta)]$ then $f_{\pm}(\cdot, \theta)$ are proportional. Then they belong to ℓ^2 and in particular are eigenvectors with eigenvalue z . Furthermore, by (1) Lemma 5.1 we have $|K(n, m)| \leq C(\theta)e^{|n-m|\Im(\theta)}$ for some constant $C(\theta)$. Then $K(n, m)$ is the kernel of an operator $K \in B(\ell^2, \ell^2)$. For any fixed m by definition of H we have

$$HA(\cdot, m)(n) = 2K(n, m) - K(n+1, m) - K(n-1, m) + q(n)KA(n, m).$$

By elementary verification for $n > m$ from the above identity we get

$$(H - z)K(\cdot, m)(n) = -((H - z)f_+)(n)f_-(m)/W(\theta) = 0,$$

while for $n < m$ we obtain similarly

$$(H - z)K(\cdot, m)(n) = -((H - z)f_-)(n)f_+(m)/W(\theta) = 0.$$

Finally, in the $n = m$ case

$$(H - z)K(\cdot, m)(m) = -((H - z)f_+)(m)\frac{f_-(m)}{W(\theta)}$$

$$+ \frac{f_+(m)f_-(m-1) - f_+(m-1)f_-(m)}{W(\theta)} = 1.$$

This implies that $(H - z)K = 1 = (H - z)R_H(z)$ and so $K(n, m) = R_H(n, m, z)$. This concludes Lemma 5.6.

Next we have, see also Theorems 1 and 2 [PS]:

Lemma 5.7. *Let $q \in \ell^{1,1}(\mathbb{Z})$.*

(a) *Assume H is generic in the sense of Lemma 5.3. Then for $\sigma > 1$ we have that for $\lambda \in \sigma_c(H)$ the following limit exists in $C^0([0, 4], B(\ell^{2,\sigma}, \ell^{2,-\sigma}))$*

$$(1) \quad \lim_{\epsilon \rightarrow 0^+} R_H(\lambda \pm i\epsilon) = R_H^\pm(\lambda).$$

Furthermore, for λ is some fixed small neighborhood in \mathbb{R} of $\sigma_c(H)$, and for $\epsilon > 0$, the operators $R_H(\lambda \pm i\epsilon)$ are Hilbert Schmidt (H -S) with H -S norm uniformly bounded.

(b) If H is not generic, then (1) exists pointwise for $\sigma > 1$ in $B(\ell^{2,\sigma}, \ell^{2,-\sigma})$ for any $0 < \lambda < 4$.

Proof. We start with (a). It is not restrictive to consider the limit from above. For $n \geq m$, by Lemma 5.6 we have for $z_\epsilon = \lambda + i\epsilon$ and for the corresponding θ_ϵ

$$\langle n \rangle^{-\sigma} \langle m \rangle^{-\sigma} R_H(n, m, z_\epsilon) = -\langle n \rangle^{-\sigma} \langle m \rangle^{-\sigma} \frac{f_+(n, \theta_\epsilon) f_-(m, \theta_\epsilon)}{[f_+(\theta_\epsilon), f_-(\theta_\epsilon)]}.$$

The fact that H is generic implies

$$[f_+(\theta_\epsilon), f_-(\theta_\epsilon)] \geq C \geq 0$$

for a fixed C . Lemma 5.1 implies for $n \geq m$ there is a fixed $C > 0$ such that $|f_+(n, \theta_\epsilon) f_-(m, \theta_\epsilon)| \leq C(1 + \max(-n, 0) + \max(m, 0))$. Then we conclude that for a fixed $C > 0$ we have

$$\begin{aligned} \sum_{m \in \mathbb{Z}} \sum_{n=m}^{\infty} \left| \langle n \rangle^{-\sigma} \langle m \rangle^{-\sigma} \frac{f_+(n, \theta_\epsilon) f_-(m, \theta_\epsilon)}{[f_+(\theta_\epsilon), f_-(\theta_\epsilon)]} \right|^2 &\lesssim \\ \sum_{n \geq m} \langle n \rangle^{-2\sigma} \langle m \rangle^{-2\sigma} (1 + \max(-n, 0) + \max(m, 0))^2 &= I_1 + I_2 + I_3, \end{aligned}$$

where I_1 involves the sum for $|n| \approx |m|$, I_2 for $|n| \gg |m|$ and I_3 for $|n| \ll |m|$. We have for $j = 1$

$$(2) \quad I_j \lesssim \sum \langle n \rangle^{-2\sigma+1} \langle m \rangle^{-2\sigma+1} < \infty.$$

We have

$$I_2 = \sum_{n \geq m, |n| \gg |m|} \langle n \rangle^{-2\sigma} \langle m \rangle^{-2\sigma} (1 + \max(-n, 0) + \max(m, 0))^2.$$

But $n \geq m$ and $|n| \gg |m|$ implies $n \geq 0$ and so we get (2) for $j = 2$. We have

$$I_3 = \sum_{n \geq m, |m| \gg |n|} \langle n \rangle^{-2\sigma} \langle m \rangle^{-2\sigma} (1 + \max(-n, 0) + \max(m, 0))^2.$$

But $n \geq m$ and $|m| \gg |n|$ implies $m \leq 0$. So we get (2) for $j = 3$.

By a similar argument

$$\sum_{n \in \mathbb{Z}} \sum_{m=n+1}^{\infty} |\langle n \rangle^{-\sigma} \langle m \rangle^{-\sigma} R_H(n, m, z_\epsilon)|^2 \leq C_\sigma < \infty.$$

So $R_H(\lambda + i\epsilon)$ are H-S, with a uniform bound on the corresponding H-S norm. By the continuous dependence of $f_{\pm}(n, \theta)$ and of the Wronskian on θ , it is elementary to conclude that the limit (1) holds in $C^0([0, 4], B(\ell^{2,\sigma}, \ell^{2,-\sigma}))$ in the generic case. By the same arguments (1) holds for $\lambda \in (0, 4)$ in the non generic case.

Notice that Lemma 5.7 (a) implies the first two claims of Lemma 3.2. The last claim of Lemma 3.2 follows from:

Lemma 5.8. *For any $u \in \ell^2(\mathbb{Z})$ we have*

$$(1) \quad P_c(H)u = \lim_{\epsilon \rightarrow 0^+} \frac{1}{2\pi i} \int_{[0,4]} [R_H(\lambda + i\epsilon) - R_H(\lambda - i\epsilon)] u d\lambda.$$

In particular, for $u \in \ell^{1,\sigma}$ with $\sigma > 1$, we have

$$(2) \quad P_c(H)u = \frac{1}{2\pi i} \int_{[0,4]} [R_H^+(\lambda) - R_H^-(\lambda)] u d\lambda.$$

Proof. (1) a consequence of the spectral theorem, see p.81 [T], while (2) holds because the right hand side of (1) converges in $\ell^{2,-\sigma}$ to the right hand side of (2).

Lemma 5.9. *Let $q \in \ell^{1,1}$ if H is generic and $q \in \ell^{1,2}$ if H is not generic. For $u \in \mathcal{S}(\mathbb{Z})$ the following are well defined:*

$$(1) \quad \begin{aligned} \int_0^4 R_H^\pm(\lambda) u d\lambda &= \int_0^\pi \sum_{\nu=-\infty}^{\infty} K_\pm(n, \nu, \theta) u(\nu) \sin \theta d\theta \\ &= \sum_{\nu=-\infty}^{\infty} \int_0^\pi K_\pm(n, \nu, \theta) \sin \theta d\theta u(\nu), \end{aligned}$$

with

$$(2) \quad \begin{aligned} K_\pm(n, \nu, \theta) &= -\frac{f_+(n, \pm\theta) f_-(\nu, \pm\theta)}{W(\pm\theta)} \quad \text{for } n \geq \nu, \\ K_\pm(n, \nu, \theta) &= -\frac{f_-(n, \pm\theta) f_+(\nu, \pm\theta)}{W(\pm\theta)} \quad \text{for } n < \nu. \end{aligned}$$

Proof. By Lemmas 5.6-7 the kernel of $R_H^\pm(\lambda)$ is given by K_\pm for $2 - 2 \cos \theta = \lambda$ with $\theta \in [0, \pi]$. The first equality in (1) is then a consequence of a change of variables. The second equality in (1) is consequence of Fubini equalities. The required summability for H generic follows from $|W(\theta)| > C > 0$ and for H non generic from the fact that $\sin \theta / W(\pm\theta)$ is a continuous function by Lemma 5.5.

We now recall Theorem 1.3:

Theorem 5.10. *We assume $q \in \ell^{1,2}$ in the non generic case and $q \in \ell^{1,1}$ in the generic case. Then we have:*

$$\|P_c(H)e^{itH} : \ell^1(\mathbb{Z}) \rightarrow \ell^\infty(\mathbb{Z})\| \leq C\langle t \rangle^{-1/3} \text{ for a fixed } C > 0.$$

Proof of generic case. It is not restrictive here to assume $n < \nu$. We have

$$\begin{aligned} P_c(H)e^{itH}(n, \nu) &= \frac{1}{2\pi i} \int_0^\pi e^{it(2-2\cos\theta)} [K_+(n, \nu, \theta) - K_-(n, \nu, \theta)] \sin\theta d\theta \\ (1) \quad &= \frac{1}{2\pi i} \int_{-\pi}^\pi e^{it(2-2\cos\theta)+i\theta(n-\nu)} K(n, \nu, \theta) d\theta \\ K(n, \nu, \theta) &:= m_-(n, \theta) m_+(\nu, \theta) \frac{\sin(\theta)}{W(\theta)}. \end{aligned}$$

For $\mathcal{M}(\mathbb{Z})$ the space of complex measures in \mathbb{Z} we have for (n, ν) fixed and taking Fourier series in θ

$$\begin{aligned} |(1)| &\leq \left\| \left[e^{it(2-2\cos\theta)} \right]^\vee \right\|_{\ell^\infty} \| [K(n, \nu, \theta)]^\wedge \|_{\mathcal{M}(\mathbb{Z})} \\ &\leq C\langle t \rangle^{-\frac{1}{3}} \| [K(n, \nu, \theta)]^\wedge \|_{\mathcal{M}(\mathbb{Z})}, \end{aligned}$$

with the second inequality due to stationary phase. We have $W^\wedge \in \ell^1$. This follows from (5.7) and Lemma 5.2. Since $W(\theta) \neq 0$ for all $\theta \in [-\pi, \pi]$, then $[1/W(\theta)]^\wedge \in \ell^1$ by Wiener's Lemma, see 11.6 [R]. By convolutions, we have $[\sin(\theta)/W(\theta)]^\wedge \in \ell^1$. If $n \leq 0 \leq \nu$ we exploit, for $\delta_{a,0}$ the Kronecker delta,

$$\|\widehat{m}_+(\nu, a) - \delta_{a,0}\|_{\ell_a^1} + \|\widehat{m}_-(n, a) - \delta_{a,0}\|_{\ell_a^1} \leq C$$

for a fixed C . Then for a fixed C we have

$$(2) \quad \| [K(n, \nu, \theta)]^\wedge \|_{\mathcal{M}(\mathbb{Z})} \leq C < \infty,$$

for all $n \leq 0 \leq \nu$. If $0 < n \leq \nu$, we can substitute $m_-(n, \theta)$ using

$$2i \sin(\theta) m_-(n, \theta) = -W(\theta) \overline{m_+(n, \theta)} + W_1(\theta) e^{-2in\theta} m_+(n, \theta).$$

and we can repeat the argument. The argument for $n \leq \nu < 0$ is similar.

Proof of non generic case. By Lemma 5.5, $\sin(\theta)/W(\theta)$ is continuous and periodic. Suppose now that $W(0) = 0$ and $W(\pm\pi) \neq 0$. We consider a smooth partition of unity $1 = \chi + \chi_1$ on $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$ with $\chi = 1$ near 0 and $\chi = 0$ near π . Then it is enough to consider (the case with χ_1 can be treated as above)

$$(3) \quad \int_{-\pi}^{\pi} e^{it(2-2\cos\theta)+i(n-\nu)\theta} K(n, \nu, \theta) \frac{\chi(\theta)}{\tilde{\chi}(\theta)} d\theta,$$

$\tilde{\chi}(\theta)$ another cutoff with $\tilde{\chi} = 1$ on the support of χ and $\tilde{\chi} = 0$ near $\pm\pi$. We set

$$\frac{1}{\sin(\theta)} = \frac{1}{2\tan(\theta/2)} + \phi(\theta), \quad \phi(\theta) := \frac{1}{\sin(\theta)} - \frac{1}{2\tan(\theta/2)}.$$

Then $\tilde{\chi}(\theta)\phi(\theta) \in C^\infty(\mathbb{T})$. Since $W^\wedge \in \ell^1$, it follows that also $[\tilde{\chi}\phi W]^\wedge(n) \in \ell^1$. If $q \in \ell^{1,2}$, we have $W^\wedge \in \ell^{1,1}$. Indeed for $m_\pm(n, \theta) = 1 + \tilde{m}_\pm(n, \theta)$ we have

$$W(\theta) = -2i\sin\theta + e^{-i\theta} (\tilde{m}_+(n+1, \theta) + \tilde{m}_-(n, \theta) + \tilde{m}_+(n+1, \theta)\tilde{m}_-(n, \theta)) - e^{i\theta} (\tilde{m}_+(n, \theta) + \tilde{m}_-(n+1, \theta) + \tilde{m}_+(n, \theta)\tilde{m}_-(n+1, \theta)).$$

Then $W^\wedge \in \ell^{1,\sigma-1}$ is a consequence of $B_\pm(n, \nu), B_\pm(n+1, \nu) \in \ell^{1,\sigma-1}$, for $\sigma = 1, 2$. This last fact for $q \in \ell^{1,\sigma}$ follows from $|B(n, \nu)| \leq e^{\gamma(n)}\eta(n + [\nu/2]) \lesssim e^{\gamma(0)}\eta(n + [\nu/2])$, proved in Lemma 5.2. Indeed

$$\sum_{\nu=1}^{\infty} \nu^{\sigma-1} |B(n, \nu)| \lesssim \sum_{\nu=1}^{\infty} \nu^{\sigma-1} \sum_{j=n+[\nu/2]}^{\infty} |q(j)| \leq \sum_{j=n}^{\infty} |q(j)| \sum_{\nu=1}^{j-n+1} \nu^{\sigma-1} \lesssim \|q\|_{\ell^{1,\sigma}}.$$

For $B_\pm(n+1, \nu)$ the argument is the same. Having established $W^\wedge \in \ell^{1,1}$, we have, see for example p.3 [Ch],

$$\left[\frac{\tilde{\chi}(\theta)W(\theta)}{-i2\tan(\theta/2)} \right]^\wedge(n) = \sum_{\nu>n} [\tilde{\chi}W]^\wedge(\nu) - \sum_{\nu< n} [\tilde{\chi}W]^\wedge(\nu).$$

Since $\tilde{\chi}(0)W(0) = 0$ and since $[\tilde{\chi}W]^\wedge \in \ell^{1,1}$, it follows $\left[\frac{\tilde{\chi}(\theta)W(\theta)}{\tan(\theta/2)} \right]^\wedge \in \ell^1$. Hence we have proved $\left[\frac{\tilde{\chi}(\theta)W(\theta)}{\sin(\theta)} \right]^\wedge \in \ell^1$. Then (3) can be bounded with the argument of the generic case. If $0 < n \leq \nu$ we can show in a similar way that $\left[\frac{\tilde{\chi}(\theta)W_1(\theta)}{\sin(\theta)} \right]^\wedge \in \ell^1$. If also $W(\pi) = 0$ we can repeat a similar argument near π .

APPENDIX A: PROOF OF LEMMA 1.1

Let us search for a solution of (1.3) in the form $u = a\varphi_0 + h$, with $h(n) \in \mathbb{R}$ for all n , $\langle h, \varphi_0 \rangle_{\ell^2} = 0$ and $a \in \mathbb{R}$. Then (1.3) becomes

$$(A.1) \quad (E_0 - \omega)a + a^7 \|\varphi_0\|_{\ell^8}^8 + \langle N(h), \varphi_0 \rangle = 0$$

$$(A.2) \quad h - R_H(-E_0)P_c(H) [(E_0 - \omega)h + (a\varphi_0 + h)^7] = 0,$$

where $N(h)(n) = \sum_{j=1}^7 \binom{7}{j} (a\varphi_0(n))^{7-j} (h(n))^j$. We have

$$R_H(-E_0)P_c(H) \in B(\ell^{p,\sigma}(\mathbb{Z}, \mathbb{R}), \ell^{p,\sigma}(\mathbb{Z}, \mathbb{R})) \text{ for any } p \in [1, \infty] \text{ and } \sigma \in \mathbb{R}.$$

The functions in (A.1)–(A.2) are C^ω in the arguments $a, \omega \in \mathbb{R}$ and $h \in \{\varphi_0\}^\perp \cap \ell^{p,\sigma}$. Substituting $h = a^7 g$ and factoring out in (A2) we obtain for g

$$(A.3) \quad g - R_H(-E_0)P_c(H) [(E_0 - \omega)g + (\varphi_0 + a^6 g)^7] = 0.$$

By the implicit function theorem applied to (A.3) we have $h = h(a, \omega) = a^7 g(a^6, \omega)$ with $g(a^6, \omega)$ real analytic in (a^6, ω) and with values in $\{\varphi_0\}^\perp \cap \ell^{p,\sigma}(\mathbb{Z}, \mathbb{R})$. Plugging in (A.1) we obtain

$$(E_0 - \omega) + a^6 \|\varphi_0\|_{\ell^8}^8 + \sum_{j=1}^7 \binom{7}{j} a^{6(j+1)} \langle \varphi_0^{7-j} g^j(a^6, \omega), \varphi_0 \rangle = 0.$$

By the implicit function theorem we obtain an analytic function $\omega - E_0 \rightarrow a^6$ with

$$a^6 = \|\varphi_0\|_{\ell^8}^{-8} (\omega - E_0) (1 + O(\omega - E_0)).$$

Then we obtain Lemma 1.1.

APPENDIX B: PROOF OF LEMMA 2.1 ON GLOBAL WELL POSEDNESS

The operator iH is a bounded skew adjoint operator in ℓ^2 and the nonlinearity $(F(u))(n) = |u(t, n)|^6 u(t, n)$ is Lipschitz continuous on bounded sets in ℓ^2 . As a consequence we have what follows.

- (1) For any $u_0 \in \ell^2$ there exist $T_1(u_0) < 0 < T_2(u_0)$ and a solution $u(t) \in C^\infty((T_1(u_0), T_2(u_0)), \ell^2)$ of (1.2) with $u(0) = u_0$. If $T_j(u_0) \in \mathbb{R}$ for a j , then

$$\lim_{t \rightarrow T_j(u_0)} \|u(t)\|_{\ell^2} = \infty.$$

For any solution $v(t)$ of the same Cauchy problem in $(\alpha, \beta) \subset (T_1(u_0), T_2(u_0))$, then $v(t) = u(t)$ in (α, β) . If $u_{0,\nu} \rightarrow u_0$ in ℓ^2 and for any bounded interval $[a, b] \subset (T_1(u_0), T_2(u_0))$, then for ν large the corresponding solutions $u_\nu(t)$ are in $C^1([a, b], \ell^2)$ and converge uniformly to $u(t)$ therein, see [CH] sections from 4.3.1 to 4.3.3. By $iu_t = -Hu - |u|^6 u$ and by the fact that the rhs is in $C^1(T_1(u_0), T_2(u_0))$ we conclude that $u \in C^2(T_1(u_0), T_2(u_0))$, and so by induction $u \in C^k(T_1(u_0), T_2(u_0))$ for all k . The continuity with respect to the initial data in $C_{loc}^k(T_1(u_0), T_2(u_0))$ is obtained similarly from the continuity in $C_{loc}^1(T_1(u_0), T_2(u_0))$.

- (2) For solutions $u(t)$ of (1.2) we have $\|u(t)\|_{\ell^2} = \|u(0)\|_{\ell^2}$. As a consequence, for any $u_0 \in \ell^2$ we have $T_2(u_0) = +\infty$ and $T_1(u_0) = -\infty$.

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